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A Borel Transform Method for Locating Singularities of Taylor and Fourier Series

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Given a Taylor series with a finite radius of convergence, its Borel transform defines an entire function. A theorem of Pólya relates the large distance behavior of the Borel transform in different directions to singularities of the original function. With the help of the new asymptotic interpolation method of van der Hoeven, we show that from the knowledge of a large number of Taylor coefficients we can identify precisely the location of such singularities, as well as their type when they are isolated. There is no risk of getting artefacts with this method, which also gives us access to some of the singularities beyond the convergence disk. The method can also be applied to Fourier series of analytic periodic functions and is here tested on various instances constructed from solutions to the Burgers equation. Large precision on scaling exponents (up to twenty accurate digits) can be achieved.

KEY WORDS: complex singularities, series analyis, nonlinear dynamics

1. INTRODUCTION

In the late nineteenth Century, Pincherle[1] and then Borel[2,3] introduced what is now known as the Borel transformation: given a formal series in powers of the complex variable Z

$$f(Z) = \sum_{n=0}^{\infty} a_n Z^n, \tag{1}$$

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one introduces the Borel transformed series

$$F(\zeta) \equiv \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n. \tag{2}$$

Since, for Re $Z > 0$,

$$\int_0^{\infty} \zeta^n e^{-Z\zeta} \, d\zeta = \frac{n!}{Z^{n+1}}, \tag{3}$$

it is useful to introduce the function

$$f^{\mathrm{BL}}(Z) \equiv \frac{1}{Z} f\left(\frac{1}{Z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{Z^{n+1}}, \tag{4}$$

which is formally the Laplace transform of $F(\zeta)$ and which in this context is sometimes called the Borel–Laplace transform of F.

Borel's motivation was predominantly to give a meaning to divergent series such as $\sum n! Z^n$ and the Borel transformation has been extensively used to resum divergent series appearing in physics (see e.g. Refs. 4–6).

In 1929 Pólya[(7)] observed that the Borel transformation can also be used to obtain information about singularities of a Taylor series (in powers of $1/Z$) with a finite radius of convergence, in which case the function $F(\zeta)$ is entire. He proved a theorem relating the convex hull of singularities of $f^{\mathrm{BL}}(Z)$ (the smallest convex set outside of which the function is analytic) to a function called the indicatrix of $F(\zeta)$, roughly the rate of exponential growth at infinity of $F(\zeta)$ as a function of the direction (for precise definitions see Sec. 4).

Here we show that this theorem can be used in conjunction with high-accuracy numerical methods to obtain very precise information on singularities of Taylor and Fourier series. Singularities play an important role in fluid dynamics and condensed matter physics (see Refs. 8–10 and references therein). Using Pólya's theorem to devise a practical numerical method would not have been possible without recent progress in high-precision numerical algorithms and, foremost, the new technique for asymptotic interpolation of van der Hoeven[(11)] which can sometimes give remarkable precision (close to twenty digits) on scaling exponents.

The paper is organized as follows. In Sec. 2 we recall some known facts about Taylor and Fourier series and their singularities. In Sec. 3 we give a presentation of van der Hoeven's method from an applied mathematics point of view and show how it works in practice, using known results for the Burgers equation. In Sec. 4 we give an elementary introduction to Pólya's theorem. In Sec. 5 we present our new method, which we propose to call BPH (Borel–Pólya–Hoeven), for determining the convex hull of singularities and, for the case of isolated singularities on this hull, their positions and type. In Sec. 6 we test BPH using again the Burgers equation. In Sec. 7 we discuss open problems and make concluding remarks.

2. FROM TAYLOR AND FOURIER COEFFICIENTS TO SINGULARITIES

We first recall the close relation between Fourier and Taylor series for analytic functions. Let $u(x)$ be a 2π-periodic function which is analytic in some neighborhood of the real axis in which it can be extended to a function $u(z)$, where $z = x + \mathrm{i}\,y$. After subtraction of a suitable constant we can assume that $\int_0^{2\pi} u(x)dx = 0$. The Fourier-series representation of u reads

$$u(x) = \sum_{k=\pm 1,\,\pm 2,\ldots} \mathrm{e}^{\mathrm{i}\,kx}\hat{u}_k, \tag{5}$$

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\,kx}u(x)\,dx. \tag{6}$$

We denote by $u^+(x)$ (resp. $u^-(x)$) the partial sum of the Fourier series (5) with $k > 0$ (resp. $k < 0$), which is analytic in the upper (resp. lower) half plane $y \geq 0$ (resp. $y \leq 0$). Each of these two functions can be written as a Taylor series by an exponential change of variable:

$$u^+(z) = \sum_{k>0} \hat{u}_k Z^k, \qquad Z \equiv \mathrm{e}^{\mathrm{i}\,z}, \tag{7}$$

$$u^-(z) = \sum_{k>0} \hat{u}_{-k} \tilde{Z}^k, \qquad \tilde{Z} \equiv \mathrm{e}^{-\mathrm{i}\,z}. \tag{8}$$

Obtaining the singularities of an analytic periodic function from its Fourier coefficients is just basically the same problem as obtaining the singularities of an analytic function $f(Z) = \sum_{n=0}^{\infty} a_n Z^n$ from its Taylor coefficients a_n. Hadamard's formula gives us the radius of convergence of the Taylor series, namely the distance to the origin of the nearest singularity(ies). If we happen to know that this is an isolated singularity at $Z_\star$, we can relate the singular behavior near $Z_\star$ to the asymptotic behavior of the a_n by the Darboux theorem.[12–14] For this one assumes that, in a neighborhood of $Z_\star$, the function $f(Z)$ is given by

$$f(Z) = (1 - Z/Z_\star)^{-\nu}\, r(Z) + a(Z), \quad \nu \neq 0, -1, -2\ \ldots, \tag{9}$$

$$r(Z) = \sum_{k=0}^{\infty} b_k\,(1 - Z/Z_\star)^k\,, \tag{10}$$

where the functions $r(Z)$ and $a(Z)$ are analytic in some disk centered at the origin with a radius exceeding $|Z_\star|$. It then follows that, for large n,

$$a_n \simeq \sum_{k=0}^{\infty} \frac{(-1)^k b_k Z_\star^{k-n}\Gamma(n+\nu-k)}{n!\,\Gamma(\nu-k)}. \tag{11}$$

The leading term is simply $a_n \simeq b_0 n^{\nu-1}/(Z_\star^n)\Gamma(\nu)$. Applied to the Fourier series (5), the leading-order Darboux formula can be recast as follows: a branch-point singularity with exponent $-\nu$ of $u(z)$ at a location $z_\star$ in the lower complex plane implies that for $k \to +\infty$ the Fourier coefficient $\hat{u}_k$ is asymptotically proportional to $k^{\nu-1}\mathrm{e}^{-\mathrm{i}kz_\star}$. This can be shown directly by applying standard steepest descent asymptotics to the integral (6).[18]

When the radius of convergence of a Taylor series is determined by a single singularity of the type assumed by Darboux, the knowledge of a sufficiently large number of Taylor coefficients with enough accuracy permits an accurate determination of the position and type of the singularity. This can be done by an iterative algorithm developed by Hunter and Guerrieri[14] or by the asymptotic interpolation method discussed in Sec. 3.

Sometimes there are two Darboux-type singularities on the convergence circle or, equivalently, the periodic function $u^+(z)$ has two singularities with the same imaginary part. The interference of the two singularities produces then a sinusoidal modulation of the Taylor coefficients. This can still be handled by an iterative algorithm,[14] but not directly by the asymptotic interpolation method, for reasons explained in Sec. 3.2. The BPH method of Sec. 5 can handle not only the case of two or more isolated singularities on the convergence circle but also "hidden" singularities located beyond this circle (or within this circle if the series is in inverse powers of Z), whose contributions to the Taylor coefficients are exponentially smaller than any term in (11). From an asymptotic point of view these contribution are "beyond all orders."

3. THE ASYMPTOTIC INTERPOLATION METHOD

Suppose that we have a function $G(r)$ of a scalar positive variable r for which we suspect that it has, for large r, an asymptotic expansion with a leading term $Cr^{-\alpha}\mathrm{e}^{-\delta r}$, as in the Darboux theorem (11), but that we only know its values numerically with high accuracy (tens to hundreds of known digits) on a regular grid $r_0, 2r_0, \ldots, Nr_0$ with a large number N of points (from fifty to thousands, depending on the problem). We set

$$G_n \equiv G(nr_0), \quad n = 1, 2, \ldots, N. \tag{12}$$

Can we determine parameters such as C, α and δ with high accuracy? One way is of course just to ignore the subleading corrections and to try a least square fit of the data to the functional form $Cr^{-\alpha}\mathrm{e}^{-\delta r}$, after taking a logarithm. One then has the awkward problem of having to pick a fitting interval of values of n; the procedure usually gives poor accuracy and the determination of subleading corrections is almost impossible.

A better way, used for example in Refs. 15–17, is to notice that, if we take the second ratio, defined as

$$R_n \equiv \frac{G_n G_{n-2}}{G_{n-1}^2} = \left(1 - \frac{1}{(n-1)^2}\right)^{-\alpha}, \tag{13}$$

then both the constant C and the exponential drop out. Assuming then n to be sufficiently large that we can ignore subleading corrections, we obtain

$$\alpha = -\frac{\ln R_n}{\ln(1 - 1/(n-1)^2)}. \tag{14}$$

The other two parameters C and δ appearing in (40) are then easily determined. If the remainder, that is the discrepancy between the value of α predicted by (14), which we denote α_n, and its limit α_∞ for $n \to \infty$, tends to zero in a known functional way, e.g., exponentially or algebraically, then we can extrapolate the α_n's to infinite values of n using, e.g., one of Wynn's algorithms[19,20] (see Ref. 21 for a review of extrapolation methods). We shall come back briefly to such issues in Sec. 3.2. Without knowing something about the functional form of the subleading corrections which control the remainder, extrapolation may not work very well because the choice of the appropriate algorithm depends on the functional form of the remainder.

Recently, van der Hoeven introduced the *asymptotic interpolation* method[11] which allows in principle the determination of the asymptotic expansion of G_n beyond leading-order terms. When the function G_n is known with very high precision and up to sufficiently large values of n, parameters such as the scaling exponent α can sometimes be determined with extreme accuracy, as we shall see in Sec. 3.1. An important feature of the asymptotic interpolation method is that it uses the determination of subleading terms to improve the accuracy on leading-order terms.

Here we shall just give a short elementary introduction to the asymptotic interpolation method for the case when the data G_n are real numbers. There are several variants of the asymptotic interpolation method; ours differs occasionally from that of Ref. 11. The basic idea of the asymptotic interpolation method is to perform simple "down" transformations on the data G_n which successively strip off leading and subleading terms. After a number of such down steps which depends on the quality of the data, the transformed data become sufficiently simple to allow a straightforward interpolation step. The list of down transformations which are needed is given hereafter.

I Inverse: $G_n \longrightarrow \frac{1}{G_n}$
R Ratio: $G_n \longrightarrow \frac{G_n}{G_{n-1}}$
SR Second ratio: $G_n \longrightarrow \frac{G_n G_{n-2}}{G_{n-1}^2}$
D Difference: $G_n \longrightarrow G_n - G_{n-1}$

At each stage, tests are applied to decide which of the four transformations should be applied in order to favor the stripping process as much as possible. If $|G_n| < 1$ for large n, apply **I**; otherwise proceed. If $|G_n|$ grows "slowly" at large n (we found that a useful operational definition is to see if the growth can be identified as algebraic with a rather well defined exponent), apply **D**; otherwise ("fast" growth), apply **R**. In addition, if $|G_n|$ grows or decreases exponentially at large n, we found that it saves time to apply **SR**; also, if $|G_n|$ is a slowly decreasing function, it is more convenient to apply $-$**D**. Note that the differences or ratios involved in **R**, **SR** and **D** are backward; this conveniently keeps the maximum index N fixed.

When the procedure is iterated, after a while, an "interpolation stage" is reached where the data can be asymptotically interpolated in a simple fashion, typically by a constant plus a small remainder tending to zero at large n. Basically this means that we have successfully stripped off a certain number of terms in the asymptotic expansion. For the kind of data which we are considering here, the most useful interpolation stages usually arise at the sixth and thirteenth stages (counting the original data as stage zero).

There are two effects which limit the number of stages which can be applied to a given set of data. First, whenever a ratio or a difference are taken, the precision of the data (i.e., the relative rounding error) deteriorates; as the number of transformations applied increases, rounding errors make the data increasingly noisy, beginning usually with the highest values of n. Second, the interpolation stages require sufficiently large values of N, since the constant asymptotic behavior at large n may be preceded by non-trivial transients. For a given resolution N and a given precision, the procedure must be stopped at the latest interpolation stage not significantly affected by the two effects just mentioned. In practice we should have a significant range of values of n over which the data are almost constant and not affected by rounding noise (if the rounding noise is very low this range may extend all the way to N). When the down process is stopped the data are interpolated and the process is reversed, by applying "up" transformations which are the inverses of the down transformation in the reverse order. The inverses of the **D**, **R** and **SR** transformations involve one or two unknown additive or multiplicative constants which are determined using the highest known values of the G_n and of their down transforms.

When the process is completed, the data are asymptotically expressed as a truncated transseries. Roughly, a transseries is a formal asymptotic series involving integer or fractional powers, logarithms, exponentials and combinations thereof.[26–28]

A worked example will now give the reader a more concrete feeling.

3.1. Testing the Asymptotic Interpolation Method on the Burgers Equation with a Single-Mode Initial Condition

Here and in Sec. 6 we shall perform tests using the one-dimensional inviscid Burgers equation

$$\partial_t u(t, x) + u(t, x)\partial_x u(t, x) = 0, \tag{15}$$

with a 2π-periodic real initial condition $u_0(x)$ having a finite number of Fourier harmonics. We begin by recalling some well-known facts about the solution and the singularities of the inviscid Burgers equation. Equation (15) has an implicit solution in Lagrangian coordinates

$$u(t, x) = u_0(a); \quad x = a + tu_0(a). \tag{16}$$

Up to the time $t_\star$ of the appearance of the first shock, the Lagrangian map $a \mapsto x$ has a Jacobian

$$J(t, a) \equiv 1 + t\partial_a u_0(a) \tag{17}$$

which does not vanish in the real space domain and (16) defines a unique real solution. This solution has singularities in the complex domain (with the real part defined modulo 2π) at locations which are the images by the Lagrangian map of the zeros of the Jacobian J. Generically these are simple zeros. The singularities in Eulerian coordinates are then square-root branch points. The solution can also be written explicitly using the Fourier–Lagrangian representation,[29,30] which in a special case was actually discovered earlier by Platzman.[31] In the periodic case, the simplest representation, called the third Fourier–Lagrangian representation, valid for $k \neq 0$, is

$$u(t, x) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\hat{u}_k(t), \tag{18}$$

$$\hat{u}_k(t) = -\frac{1}{2\mathrm{i}\pi kt}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k(a+tu_0(a))}da. \tag{19}$$

In this section we take the "single-mode" initial condition

$$u_0(a) = -\frac{1}{2}\sin a, \tag{20}$$

for which the first real singularity is at $t_\star = 2$. Using (19) and the integral representation of the Bessel function J_n of integer order n (see, e.g., Ref. 32 p. 360), one finds[31]

$$\hat{u}_k(t) = \frac{\mathrm{i}}{kt}J_k(kt/2). \tag{21}$$

For convenience, we shall consider the solution at $t = 1$. This single-mode solution has only one pair of complex conjugate singularities on the imaginary axis at

$$z_\star^\pm = \pm \mathrm{i}\,\delta, \qquad \delta = \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}. \tag{22}$$

Bessel functions of large order and arguments have an asymptotic expansion (in the sense of Poincaré), obtained through the method of steepest descent by Debye.[33] (The matter is also discussed in Chap. VIII of Ref. 34). Debye identified various asymptotic regimes which, in our notation, depend on whether t is less or larger than $t_\star$ and is or is not very close to $t_\star$; his classification is in one-to-one correspondence with that of the various regimes relating to preshocks, as discussed, for example, in Refs. 29–30. When $t = 1$, well before $t_\star$, the relevant Debye expansion for $k \to +\infty$ is:

$$\hat{u}_k(1) \simeq \frac{\mathrm{i}}{\sqrt{\pi\sqrt{3}}} k^{-\frac{3}{2}} \mathrm{e}^{-\delta k} \left(1 + \sum_{n=1}^{\infty} \frac{\gamma_n(2/\sqrt{3})}{k^n} \right), \tag{23}$$

where δ is given by (22),

$$\begin{aligned} \gamma_1(\xi) &= \frac{3\xi - 5\xi^3}{24}, \\ \gamma_2(\xi) &= \frac{81\xi^2 - 462\xi^4 + 385\xi^6}{1152}, \\ \gamma_3(\xi) &= \frac{30375\xi^3 - 369603\xi^5 + 765765\xi^7 - 425425\xi^9}{414720}, \end{aligned} \tag{24}$$

and the higher-order polynomials $\gamma_n(\xi)$ satisfy recurrence relations given, e.g., in Ref. 32. The leading term of this expansion follows also from the Darboux theorem (11).

Let us now show that the asymptotic interpolation method, as outlined in Sec. 3, when applied to the Fourier coefficients of the single-mode solution (21) can recover a suitably truncated version of the Debye expansion (23). We use all the Fourier coefficients with $k = 1, \ldots, N$, where $N = 1000$ and define our initial data set as $G_n \equiv \hat{u}_n(1)/\mathrm{i}$.

Each coefficient is calculated with an 80-digit precision (using Mathematica® and 120-digit working precision). The basic transformations and their inverses are implemented numerically in 80-digit precision, using the high-precision packages GMP and MPFR available from `http://www.swox.com/gmp/` and `http://www.mpfr.org/`.

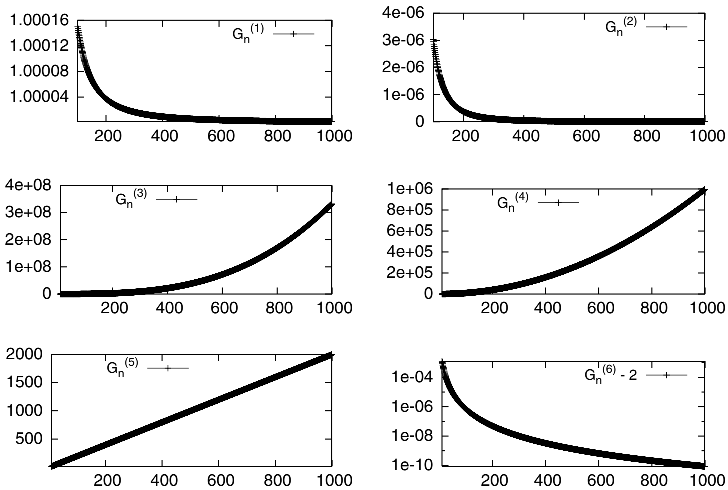

Fig. 1. Numerical output from asymptotic interpolation at stages 1–6. Stages 1–5 are represented in linear coordinates. For stage 6 we represent the difference between $G_6(n)$ and its asymptotic value 2 in lin-log coordinates.

With these data we are able to reach stage 13. The list of successively applied transformations, resulting from the tests given in Sec. 3, is

$$\textbf{SR, -D, I, D, D, D, D, I, D, D, D, D, D} \tag{25}$$

Figure 1 shows the first six stages. It is mostly intended to bring out overall features and to make clear which of the four transformations is to be selected at the next stage

It is very easy to understand why the first six stages are as listed above. Indeed, let us suppose that, to leading order, $G_n = Cn^{-\alpha}\mathrm{e}^{-\delta n}$. We can work out analytically the various transforms and we list hereafter the result up to the sixth stage, displaying only the leading and when needed the first subleading term in the large-n expansion:

$$Cn^{-\alpha}\mathrm{e}^{-\delta n} \xrightarrow{\mathbf{SR}} 1+\frac{\alpha}{n^2} \xrightarrow{\mathbf{-D}} \frac{2\alpha}{n^3} \xrightarrow{\mathbf{I}} \frac{n^3}{2\alpha} \xrightarrow{\mathbf{D}} \frac{3n^2}{2\alpha} \xrightarrow{\mathbf{D}} \frac{3n}{\alpha} \xrightarrow{\mathbf{D}} \frac{3}{\alpha} \tag{26}$$

It is seen that stages 1 and 6 are interpolation stages at which the data are asymptotically flat. Stage 6 is particularly important since the asymptotic value $3/\alpha$ gives the scaling exponent α. According to (23), for the Burgers single-mode solution, the asymptotic value should be 2.

Let us now show in some detail how the asymptotic interpolation technique works to give us the asymptotic expansion of G_n. We begin by limiting ourselves to a six-stage procedure. The successively transformed data will be denoted

$G^{(1)}, \ldots, G^{(6)}$. Following Ref. 11, we interpolate $G^{(6)}$ by $3/\alpha$. How cleanly this can be done is visible in the last of the graphs in Fig. 1, where we show the discrepancy between $G_n^{(6)}$ and its asymptotic value 2. This discrepancy falls to about 10^{-10} at the upper end of the range. Then we determine $G^{(5)}$ by inverting the relation $G^{(6)} = \mathbf{D}G.^{(5)}$ This involves an unknown additive constant which is determined from the last data point $G_N^{(5)}$. Then we continue inverting the $\mathbf{D}$ operators appearing at stages 4 and 5, each time using the last point to obtain the additive constant. In this way we obtain a cubic polynomial for $G.^{(3)}$ We then invert the operator $\mathbf{I}$ and obtain the inverse of the aforementioned cubic polynomial, which can be written $-2\alpha n^{-3}(1 + d_1/n + d_2/n^2 + d_3/n^3 + \cdots)$ with in principle well defined constants d_1, d_2, etc. Then we invert the operator $\mathbf{SR}$; this can be done by taking a logarithm which will transform second ratios into second increments. At the end of the process we obtain the asymptotic expansion

$$G_n \simeq Cn^{-\alpha}\mathrm{e}^{-\delta n}\left[1 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \frac{\gamma_3}{n^3} + O\left(\frac{1}{n^4}\right)\right]. \tag{27}$$

It is actually simpler to start with (27) and to apply successively the first six transformations listed in (25) to identify the parameters. With the six-stage procedure we obtain $C, \alpha, \delta, \gamma_1, \gamma_2, \gamma_3$. Their values are given in Table 1.

It is seen that the coefficients C, α and δ appearing in the leading term have a precision of at least 10^{-10}. The precision of the coefficients γ_i for the subleading terms deteriorates with the order.

We now turn to the analysis using a 13-stage procedure. This allows a much more precise determination of the aforementionned coefficients and, in principle, the determination of six additional terms in the expansion (27). After stage 6, the

Table I. Solution to the Burgers equation with single-mode initial condition: Comparison of theoretical values with 6-stage and 13-stage asymptotic interpolation values for the first six coefficients in Debye's solution (23).

	α	δ	C
6 stages	1.49999999993	0.4509324931404	0.4286913791
13 stages	1.49999999999999995	0.45093249314037806 1868	0.4286913790524959
Theor. value	3/2	0.450932493140378061861	0.428691379052495856 43
	γ_1	γ_2	γ_3
6 stages	−0.17641252	0.17295	−0.401
13 stages	−0.17641258225238	0.172968106990	−0.406446182
Theor. value	−0.176412582252385	0.1729681069958	−0.4064461802
	γ_4	γ_5	γ_6
13 stages	1.384160933	−6.192505762	34.5269751
Theor. value	1.3841609326	−6.1925057618568063655	34.526975286449930956

Note. For 13-stage asymptotic interpolation we also give some of the higher order coefficients.

next interpolation stage is stage 13. This is easily shown by observing that the discrepancy between $G_n^{(6)}$ and its asymptotic value $3/\alpha = 2$ is $O(1/n^4)$, because all the lower-order terms have been stripped off by the first six transformations. More specifically, we have

$$G_n^{(6)} = 2 + \frac{c_1}{n^4} + \frac{c_2}{n^5} + \frac{c_3}{n^6} + \frac{c_4}{n^7} + \frac{c_5}{n^8} + \frac{c_6}{n^9} + \ell_n, \tag{28}$$

$$\ell_n = O\left(\frac{1}{n^{10}}\right). \tag{29}$$

Stages 7–13 gives us the coefficients $c_1, \ldots, c_6$ and allow us to find the remainder r_n, as defined in (28). We found that r_n, determined by this 13-stage procedure, falls to about 5×10^{-17} at the end of the range. As shown in Table 1, the precision on the first six coefficients in the asymptotic expansion has improved very much and is now of a few 10^{-17} for the exponent α.

3.2. Further Remarks on Asymptotic Interpolation

The method of asymptotic interpolation is still in the development stage; improvements and new features are thus to be expected. Some are already suggested in the initial publication.[11] One rather straightforward extension is from real to complex data. For rapidly growing data, one can use logarithms instead of ratios. In Ref. 11 it is recommended to take ratios or logarithms as often as possible and to define "slow growth" as slower than, say, $n^{5/2}$. This helps in identifying the functional form of the transseries expansion. Once this is known, we found that the values of the coefficients can be generated more efficiently by using a rather broad definition of slow growth, namely well-identifiable polynomial behavior.

A very important issue is to determine how many stages are feasible with a given resolution N and a given precision. We have found that the successive interpolation stages, at which the data are asymptotically flat, have this flat regime preceded by longer and longer transients. To make this more concrete we have investigated how far it is necessary to go to be within five per cent of the asymptotic value for the Burgers single-mode problem. For stage 6 the asymptotic value is 2 and the data are within five per cent everywhere. For stage 13, the asymptotic value is about 0.33836513 and less than five per cent discrepancy holds for $n > 25$. The next interpolation stage has number 20. The asymptotic value is $2/7$ but less than five per discrepancy holds only beyond $n = 1620$. In Fig. 2 we have represented the data at stage 20. It is seen that the discrepancy is enormous until we reach well beyond $n = 1000$. Obviously, if N is not large enough the stripping of subleading terms performed by the successive stages must be stopped *however high the precision of the data may be*. A related issue is discussed at the end of Sec. 7 of Ref. 11.

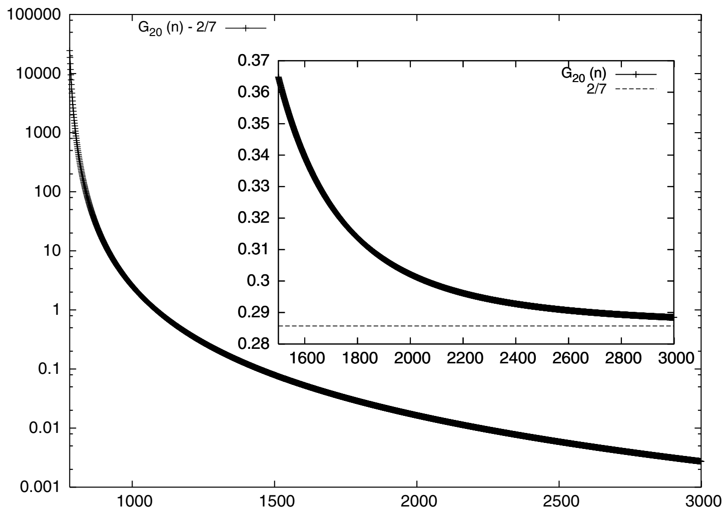

Fig. 2. Interpolation stage 20 has the asymptotic value $2/7$ as shown in the inset. The main figure shows the discrepancy $G_{20}(n) - 2/7$.

Since rounding errors increase with the stage number, a certain balance must be kept between resolution and precision. To investigate this quantitatively on the Burgers single-mode problem, we have artificially degraded our precision by adding random noise of various strengths. It appears that we need at least 16 significant digits at stage 6 and 27–35 significant digits at stage 13.

If we are only interested in obtaining an accurate determination of a few terms in the expansion (27), we may be able to retrieve them using asymptotic interpolation stopped at the sixth stage and continuing with a different strategy. Indeed we observe that, at the sixth stage, the data given by (29) have the form

$$G_n^{(6)} = s + r_n, \tag{30}$$

where the remainder r_n decays to zero as $n \to \infty$. This is a well studied situation in the theory of convergence acceleration by sequence transformations, whose goal is to replace the sequence (30) by a transformed sequence having the same limit s but a much faster decaying remainder (see, e.g., Refs. 21, 22). A simple and very popular acceleration method appropriate for (29) is Wynn's rho-algorithm,[(19)] although more sophisticated methods are known.[(21)] In our case it gives the correct value 2 with a 20-digits precision. This is even better than the 13-stage asymptotic interpolation. Note that the choice of a particular convergence acceleration method depends crucially on the functional form of the remainder. With asymptotic interpolation this form can be determined rather than having to be assumed. Here a caveat is in order: if the data are not sufficiently asymptotic the mixed procedure just described will not work, for example because the remainder has not yet settled down to algebraic decrease. A situation of this type seems to be present in the work on short-time asymptotics discussed in Ref. 17: the rho-algorithm does not improve the quality of the scaling exponent controlling the divergence of the vorticity at the singular manifold and much higher resolutions are probably needed for that problem.

E.J. Weniger (private communication) has pointed out that asymptotic interpolation and sequence transformations have technical features in common. In asymptotic interpolation one tries to annihilate leading terms in the asymptotic expansion, whereas in sequence transformation one tries to shrink the remainder by annihilating its largest contributions, but the transformations used in both instance are often the same, for example, finite difference operators.

A powerful method of asymptotic series analysis, widely applied in statistical physics, is the method of differential approximants, which can be viewed as a generalization of the Dlog Padé method.[9] In particular it has been used to analyze self-avoiding walks (SAWs) and polygons (SAPs). We have applied the asymptotic interpolation method to data available at `http://www.ms.unimelb.edu.au/~iwan/polygons/Polygons_ser.html`. The goal was to see how well we can reproduce the asymptotics of the number of self-avoiding polygons with $2n$ steps on square and honeycomb lattices.[23–25] When analyzing the square lattice data for the largest available range, that is n up to 55, we found that asymptotic interpolation gives the value of the critical point correct to 9 decimal places, whereas differential approximants give about 3 additional digits. We observe that (i) the actual implementation with asymptotic interpolation is somewhat simpler and (ii) asymptotic interpolation is not limited to problems which can be well approximated by solutions of low-order linear differential equations.

The method of asymptotic interpolation is, in our opinion, very useful but is of course not the *panacea*. One disease it cannot directly cure is the presence of sinusoidal oscillations. For example if the analytic function $f(Z)$ has two complex conjugate singularities at $R\mathrm{e}^{\pm \mathrm{i}\phi_\star}$ on its circle of convergence, large-order Taylor coefficients will present a sinusoidal oscillation with a wavelength proportional to $\phi_\star$. After any number of stages, this oscillation is still present and the data cannot be interpolated by a constant. As we shall see now, a Borel transformation takes care of this problem and can also bring hidden singularities to the foreground.

4. PÓLYA'S THEOREM

Here we just want to give the reader a good feeling of what the theorem states and a heuristic derivation. We begin with examples discussed in Sec. 32 of Ref. 7.

Let $c = |c|\mathrm{e}^{-\mathrm{i}\gamma}$ be a complex number and consider the function

$$F(\zeta) = \mathrm{e}^{c\,\zeta} = 1 + \frac{c\,\zeta}{1!} + \frac{c^2\zeta^2}{2!} + \cdots, \tag{31}$$

which corresponds to the choice $a_n = c^n$ in (2). The Borel–Laplace transform, given by (4), is

$$f^{\mathrm{BL}}(Z) = \frac{1}{Z} + \frac{c}{Z^2} + \frac{c^2}{Z^3} + \cdots = \frac{1}{Z-c}. \tag{32}$$

It has a pole at $Z = c$, whereas $F(\zeta)$ is an entire function (analytic in the whole complex domain). We set $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$ and let $r \to \infty$, holding the direction ϕ fixed; the modulus of $F(\zeta) = \mathrm{e}^{|c|r\mathrm{e}^{\phi-\gamma}}$, in the direction ϕ, varies exponentially at the rate $h(\phi) = |c|\cos(\phi - \gamma)$, called the *indicatrix* of F. We define $k(\phi) \equiv \mathrm{Re}\,(c\,\mathrm{e}^{-\mathrm{i}\phi}) = |c|\cos(\phi + \gamma)$. This is the (signed) distance of the origin to the line normal to the direction ϕ passing through the pole c and is called the *supporting function* of the (single) singularity. We observe that

$$h(\phi) = k(-\phi). \tag{33}$$

This relation is the simplest instance of Pólya's theorem.

Next, following again Pólya's Sec. 32, we want to have n distinct poles at the complex locations $c_1,\ c_2, \ldots,\ c_p$. For this we take complex linear combinations with non-vanishing coefficients $C_1,\ C_2, \ldots,\ C_p$:

$$F(\zeta) = C_1\mathrm{e}^{c_1\zeta} + C_2\mathrm{e}^{c_2\zeta} + \cdots + C_p\mathrm{e}^{c_p\zeta}. \tag{34}$$

The Borel–Laplace transform is

$$f^{\mathrm{BL}}(Z) = \frac{C_1}{Z-c_1} + \frac{C_2}{Z-c_2} + \cdots + \frac{C_p}{Z-c_p}. \tag{35}$$

For any $\phi \in [0, 2\pi]$, we define now the indicatrix and the support function, a little more formally, as

$$h(\phi) \equiv \limsup_{r\to\infty} r^{-1} \ln|F(r\mathrm{e}^{\mathrm{i}\phi})|, \tag{36}$$

(in the present example, the lim sup is just an ordinary limit) and

$$k(\phi) \equiv \sup_{z\in K}(X\cos\phi + Y\sin\phi) = \sup_{Z\in K}\{\mathrm{Re}\,(Z\mathrm{e}^{-\mathrm{i}\phi})\}, \tag{37}$$

where $Z = X + \mathrm{i}\,Y$ and $K \equiv \{c_1,\ c_2, \ldots, c_p\}$ is the singular set. Since there is a finite number of singularities, the sup operation is just the same as the maximum. We define a *supporting line* of K as a line which has at least one point in common with K and such that all the points of K are in the same half space with respect to the line. The intersection of all these half spaces is the *convex*

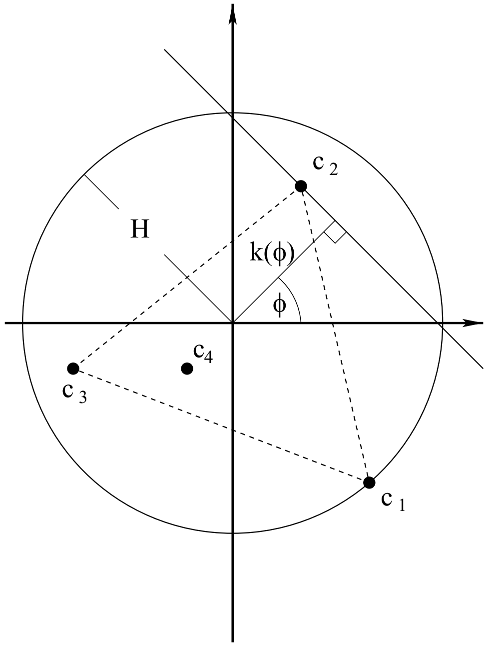

Fig. 3. Construction of the supporting function $k(\phi)$ of the set K of singularities of $f^{\mathrm{BL}}(Z)$. The singularity c_1 is on the convergence circle; the convex hull of K is defined by c_1, c_2 and c_3. The singularity c_4 is inside the convex hull.

hull of K. In the present case this is just the smallest convex polygon containing all the poles. It is readily seen that $k(\phi)$ is the (signed) distance of the origin to the supporting line normal to the direction ϕ (see Fig. 3.) The rate of growth of $F(\zeta)$ in the direction ϕ is obviously that of the fastest growing of the p exponentials in (34), which is precisely $k(-\phi)$, so that Pólya's relation (33) holds.

He proved a much more general theorem: *Let $f^{\mathrm{BL}}(Z)$ be an analytic function defined by the Taylor series (4) in powers of 1/Z which has a finite non-vanishing radius of convergence H, and let K be the smallest convex compact set containing its singularities, then (i) the Borel transformed series (2) defines an entire function of exponential type, and (ii) the indicatrix $h(\phi)$ of $F(\zeta)$, defined by (36), and the supporting function $k(\phi)$ of K, defined by (37), are related by $h(\phi) = k(-\phi)$ and $H = \sup_\phi h(\phi)$.*

(An entire function $F(\zeta)$ is said to be of exponential type if its modulus is bounded by $A\mathrm{e}^{a|\zeta|}$, where A and a are suitable positive constants.)

Pólya's proof (not given here) makes use of the fact that f^{BL} and F are Laplace transformed of each other, specifically,

$$f^{\mathrm{BL}}(Z) = \int_0^{\infty} F(\zeta)\mathrm{e}^{-Z\zeta}\, d\zeta, \tag{38}$$

$$F(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} f^{\mathrm{BL}}(Z)\mathrm{e}^{Z\zeta}\, dZ, \tag{39}$$

where a is any real number such that the singular set K is entirely contained in $\mathrm{Re}\, Z < a$.

Observe that no particular assumption is made regarding the type of the singularities which can be isolated (e.g. poles or branch points) or continuously distributed (natural boundary). Inside the circle of convergence $|Z| = H$, the series (4) is divergent. However if the whole circle is not a natural boundary, the function $f^{\mathrm{BL}}(Z)$ can be analytically continued to suitable Z's inside this circle and the pair of integrals (38) and (39)can be viewed as a way of resumming the divergent series (4).

In applications it frequently happens that all the "edge singularities," that is, those determining the border of the convex set K are isolated. This border is then piecewise linear, as in the case of n poles discussed above. The angular dependence of the supporting function is then given by $k(\phi) = |c_j| \cos(\phi + \gamma_j)$ in the angular interval $\phi_{j-1} < \phi < \phi_j$ for which the supporting line normal to ϕ touches K at $c_j = |c_j|\mathrm{e}^{-\mathrm{i}\gamma_j}$ (see Fig. 3). If $k(\phi)$ is known with high accuracy, then the positions of the edge singularities c_js can also be determined accurately.

Moreover we can then determine the type of an isolated singularity at c_j by studying the asymptotic behavior of $F(\zeta)$ along rays $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$ with large r, in a suitable angular interval. For example, let us assume that, near c_j the function $f(z)$ has an algebraic singularity and is to leading order proportional to $(Z - c_j)^{\alpha-1}$, where the exponent α is real and not a positive integer. (If $\alpha - 1 > 0$, this behavior is assumed for a suitable first- or higher-order increment of f.) After shifting the contour of integration to follow the boundary of K near c_j (cf. Fig. 4), application of steepest descent[37] to (39) with $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$ taken in the angular sector $\phi_{j-1} < -\phi < \phi_j$ and $r \to \infty$ yields

$$G(r) = \left|F(r\mathrm{e}^{\mathrm{i}\phi})\right| = Cr^{-\alpha}\mathrm{e}^{h(\phi)r}[1 + \varepsilon(r)], \tag{40}$$

$$h(\phi) = |c_j| \cos(\phi - \gamma_j). \tag{41}$$

Here C is a positive constant and $\varepsilon(r)$ tends to zero for $r \to \infty$ at a rate which depends on what is assumed for subleading corrections to the $(z - c_j)^{\alpha-1}$ singular behavior. If we are able to identify the algebraic prefactor to the exponential in (40), we can find the exponent α of the algebraic singularity.

Non-algebraic singularities can be handled similarly. For example, if near c_j the function $f(z)$ behaves as $\mathrm{e}^{1/(Z-c_j)}$, application of steepest descent shows that instead of the algebraic prefactor proportional to $r^{-\alpha}$ which appears in (40), we

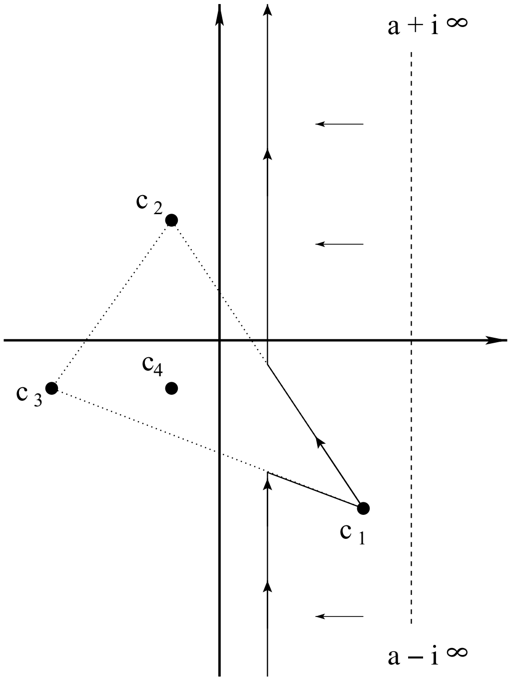

Fig. 4. Contour of integration for computing the inverse Laplace transform of $f^{\mathrm{BL}}(Z)$ (*dashed line*) and its deformation to obtain the asymptotic contribution from the singularity at c_1 (*continuous line with arrows*).

obtain an exponential prefactor proportional to $\mathrm{e}^{\pm 2\cos(\phi/2)\sqrt{r}}$. Furthermore, if all the singularities on the convex hull of K are isolated, then (37) remains valid: the indicatrix is piecewise a cosine function.

How this is done in practice will be discussed in the next section.

5. THE BOREL–PÓLYA–HOEVEN METHOD

As we have seen in Sec. 3, the asymptotic interpolation method, applied to the Taylor coefficient of an analytic function with a finite radius of convergence determined by a single isolated singularity allows one to identify its location and type. This is not the case if there is more than one singularity on the convergence circle. Furthermore "hidden" singularities are not directly retrievable from the asymptotics of Taylor coefficients.

We can however take advantage of Pólya's theorem (Sec. 4) to replace the analysis of large-order Taylor coefficients of an analytic function by the analysis of the behavior of its Borel transform at large distances in the complex ζ plane along various rays. This behavior can be found by asymptotic interpolation, from which we can then construct the convex hull K of the singularities and obtain their type when they are isolated. This is the BPH strategy which we now describe in a more detailed way.

We start from a truncated Taylor series in inverse powers of Z

$$f_{\mathrm{T}}^{\mathrm{BL}}(Z) = \sum_{n=0}^{N} \frac{a_n}{Z^{n+1}}, \tag{42}$$

with N terms, each of the coefficients being known with a precision ε. We construct the associated truncated Borel series

$$F_{\mathrm{T}}(\zeta) \equiv \sum_{n=0}^{N} \frac{a_n}{n!}\zeta^n. \tag{43}$$

We choose a certain number of discrete angular directions characterized by their angle ϕ. Along each ray $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$, we evaluate the modulus of the Borel series at M points spaced by a constant distance (mesh) r_0:

$$G_m \equiv |F_{\mathrm{T}}(mr_0\mathrm{e}^{\mathrm{i}\phi})|, \quad m = 1, 2, \ldots, M. \tag{44}$$

Then we apply the asymptotic interpolation method of Sec. 3 to identify a large-m leading-order behavior. For example, for algebraic singularities we have

$$G_m \simeq C(\phi)(mr_0)^{-\alpha(\phi)}\mathrm{e}^{h(\phi)mr_0}. \tag{45}$$

This gives us the constant $C(\phi)$, the prefactor exponent $-\alpha(\phi)$ and the indicatrix $h(\phi)$ for the discrete set of directions. Pólya's theorem then gives us the supporting function $k(\phi) = h(-\phi)$ of the set K of singularities of the Taylor series. As we have seen in Sec. 4, if the singularities on the convex hull of K are isolated and are located at $|c_j|\mathrm{e}^{-\mathrm{i}\gamma_j}$, then the supporting function is piecewise a cosine function, given by $|c_j|\cos(\phi + \gamma_j)$. The exponent α gives us the type of the singularity: a branch point (or a pole) of exponent $\alpha - 1$. Other types of singularities, for example of the exponential type discussed near the end of Sec. 5, are handled similarly after identification of the appropriate asymptotic behavior.

In practice, we have to choose the set of discrete directions, the mesh r_0 and the maximum number of points M on each ray. If we happen to know the number p of isolated singularities and, at least approximately, their positions we can pinpoint the latter by taking $2p$ suitable ϕ directions. This is however rarely the case. We recommend taking a fairly large set of directions (for example 500 uniformly spaced directions) in order to reduce the risk of missing one or several of the cosine functions. The natural choice for the mesh r_0 is H^{-1} where H is the radius of convergence of the Taylor series. An approximate value is $H_{\mathrm{approx}} = (1/n)\ln|a_n|$ for large n, which is roughly constant. For the determination of the largest distance $r_{\max} = Mr_0$ we limit ourselves to the case where the function $F(r\mathrm{e}^{\mathrm{i}\phi})$ grows at large distances, that is $h(\phi) > 0$ (otherwise there are severe numerical problems). $r_{\max}$ is then determined by the condition that the last term $a_N\zeta^N/N!$ of the (truncated) Borel series (43) should introduce a relative error in the determination of $F(r\mathrm{e}^{\mathrm{i}\phi})$ which does not exceed the precision ε with which the Taylor coefficients are known. A rough estimate for $|a_N|$ is H^N and for $|F(r_{\max}\mathrm{e}^{\mathrm{i}\phi})$ is $\mathrm{e}^{r_{\max}H}$. Using the Stirling formula, we find that, to leading order $M \simeq N$ (the dependence on ε appears only in subleading logarithmic corrections).

We mention that an improvement would be to replace a mere polynomial truncation of the Borel series by a suitable resummation/acceleration method for computing entire functions.[38] This could be crucial for determining negative indicatrix values, that is, when $F(\zeta)$ is exponentially decreasing at large ζ.

It is of interest to know how well we can separate two discrete singularities. By Pólya's theory, each singularity contributes an exponential term to the large-r behavior of the modulus of the Borel transform. If r becomes sufficiently large compared to the difference in the two e-folding rates, only one of the two singularities will be seen. By suitably changing the direction of the ray in the ζ-plane we can then focus separately on each singularity. The worst case for discrimination is when we have two singularities which are at the same distance of the origin. Assuming that this common distance is comparable to the radius of convergence H and denoting by Δ the distance of the two singularities, we find that the largest discrepancy in e-folding rate is roughly Δ^2/H. Denoting, as above, by M the maximum number of point on a ray, we find that good separation requires the separation parameter $M\Delta^2/H^2$ to be large. Since discrepancies are amplified exponentially, a separation parameter of 10 may suffice.

We shall not here discuss issues of algorithmic complexity, such as the reduction of the number of operations to evaluate the truncated Borel series. In applications the complexity of the numerical calculations needed to accurately determine the Taylor coefficients will usually exceed very much what is needed for the BPH analysis.

6. TESTING BPH ON THE BURGERS EQUATION WITH MULTIMODE INITIAL CONDITIONS

To test the BPH method we need a Taylor series having either a pair of singularities on the convergence circle or "hidden singularities." As in Sec. 3.1, this can be done using 2π-periodic solutions of the inviscid Burgers equation. The 2-mode initial condition

$$\begin{aligned} u_0(a) &= \lambda_1 \sin a + \lambda_2 \sin(2a), \\ \lambda_1 &= -1/2, \quad \lambda_2 = (1/16)(4 - \sqrt{(14)}) + \epsilon, \\ \epsilon &= 1/150, \end{aligned} \tag{46}$$

produces at $t = 1$ a solution $u(1, z)$ having, in Eulerian coordinates, singularities at

$$\begin{aligned} z_\star^\pm &= \pm 0.1103542160016972443 \\ &\quad \pm \mathrm{i}\ 0.737097018253664793. \end{aligned} \tag{47}$$

Henceforth we shall concentrate on the singularities of $u(1, z)$ in the lower half plane, which are also the singularities of the function $u^+(1, z)$, the sum of Fourier harmonics with $k > 0$. Note that there are two singularities with the same imaginary part and opposite real parts (this is a consequence of the symmetry $a \mapsto -a$, $u_0 \mapsto -u_0$ of the initial condition and of the complex conjugate symmetry). When the Fourier series for $u^+(1, z)$ is transformed into a Taylor series in inverse powers of Z by setting $Z = \mathrm{e}^{-\mathrm{i}z}$, the z singularities get mapped onto two complex conjugate Z singularities

$$Z_\star^\pm = 0.4755903313336372343 \pm \mathrm{i}\ 0.0526974896343733942. \tag{48}$$

The 3-mode initial condition

$$u_0(x) = \lambda_1 \sin x + \lambda_2 \sin 2x + \lambda_3 \sin 3x \tag{49}$$

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}, \quad \lambda_2 = \frac{4 - \sqrt{14}}{16} + \frac{1}{50}, \\ \lambda_3 &= -\frac{1}{100}, \end{aligned} \tag{50}$$

produces at $t = 1$ a solution $u(1, z)$ having, in Eulerian coordinates, in the lower complex half plane singularities at

$$z_{1\star} = -\mathrm{i}\ 0.4608974136239120258 \tag{51}$$

$$\begin{aligned} z_{2\star}^\pm &= \pm 0.8575677577466957833 \\ &\quad -\mathrm{i}\ 1.1175132271503113898. \end{aligned} \tag{52}$$

The $z_{1\star}$ singularity is on the imaginary axis and is the closest to the real domain. The other two are further away (hidden) and symmetrically located with respect to the imaginary axis. The corresponding Z singularities are

$$Z_{1\star} = 0.6307173770893952917, \tag{53}$$

$$\begin{aligned} Z_{2\star}^\pm &= 0.2140094820693456182 \\ &\quad \pm \mathrm{i}\ 0.2473645913888956747. \end{aligned} \tag{54}$$

are shown in Fig. 5.

To apply the BPH method we generate the Fourier harmonics with $k = 1, \ldots, N = 1000$ using the third Fourier–Lagrangian representation (19). The

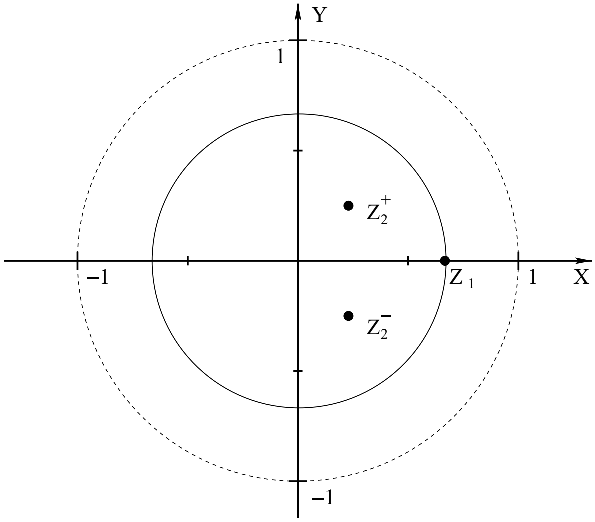

Fig. 5. Positions in Z-plane of the singularities for the 3-mode initial condition. Continuous line: circle of convergence; dashed line: image of the real domain by the map $z \mapsto \mathrm{e}^{-\mathrm{i}z}$.

Lagrangian integrals are again calculated with 80-digit precision and 120-digit working precision. The Borel transform is calculated for 20 values of ϕ between 0 and $\pi/2$ (a symmetry $\phi \to -\phi$ makes it unnecessary to take $\phi < 0$ and for $\phi > \pi/2$ the indicatrix is negative). For the mesh we take $r_0 = 1$. The total number of points on each ray is $M = 500$. Along each selected ray, application of six-stage asymptotic interpolation gives us C, α and h. We can check that the indicatrix is piecewise a cosine function as implied by (37). Least square fits allow us to identify the parameters of these cosine functions and thereby to find the locations of the singularities. We recover the known values with an accuracy of about 10^{-6}. We found that the accuracy on "hidden singularities" is comparable to that on directly visible ones. We also found that $N = 1000$ is not sufficiently asymptotic for a 13-stage analysis of the kind described in Sec. 3.1.

7. CONCLUDING REMARKS

One central theme of this paper is the use of a Borel transform, in conjunction with Pólya's theorem, to reveal singularities not directly accessible from the asymptotic behavior of the Taylor/Fourier coefficients. A very useful property of the Borel transform of a Taylor series (in inverse powers of Z) is that its large-distance behavior encodes information not only about those singularities of the Taylor series located on its convergence circle, but also about other singularities "hidden" inside this circle. Actually the Borel transform, followed by a

Borel–Laplace transformation is a way of performing analytic continuation. Recovering hidden singularities from a Taylor series has important applications in a number of fields; many of the known techniques have been reviewed by Guttmann.[9]

To the best of our knowledge Pólya's theorem has never been used as a numerical tool for identifying singularities. The theorem is of a very general nature and assumes nothing about the nature of the singularities; this has the great advantage that we do not have to distinguish between true and spurious singularities, as is the case, for example, when using Padé approximants and related methods. The principal drawbacks are that (i) not all hidden singularities are accessible, only those located on the convex hull of the singular set, (ii) pairs or clusters of singularities situated too close to each other may not be easily distinguishable, and (iii) enough terms in the series must be known to be able to actually obtain the asymptotic behavior of the Borel transform. When hundreds to thousands of Taylor coefficients are known, alternative mathematically well-founded techniques may become competitive, for example the old Weierstrass analytic continuation method; thanks to recent algorithmic discoveries it can be performed quite efficiently.[39,40]

The other theme of this paper is the asymptotic interpolation method of van der Hoeven which is here used both directly (when Darboux's theorem is applicable) and indirectly by means of Pólya's theorem. When a large number of Taylor/Fourier coefficients are known with sufficient accuracy, asymptotic interpolation can give truly remarkable results, providing us not only with very accurate leading terms but also with several subleading corrections. As we have seen in Sec. 3, there is usually a well-defined relation between the number of subleading correction terms and the number of stages of the procedure which can be achieved. The latter depends crucially on the number of known coefficients and on their precision. For example, if the data have only double precision, it is unlikely that more than six stages can be achieved. Asymptotic interpolation might than be viewed as an overkill compared to more standard techniques, but it is worth stressing that asymptotic interpolation is very easy to implement.

Which kinds of problems are most likely to fall within the prongs of full-strength Borel–Pólya–Hoeven-type analysis? This depends crucially on the computational complexity of the problem, that is the dependence of CPU requirement and storage on the number of coefficients N. As pointed out by Guttmann,[9] phase transition problems formulated on a lattice require usually enumerating diagrams and the number of these tends to grow exponentially with order, while fluid dynamics problems generally have only polynomial complexity. In connection with phase transitions our BPH method is likely to be less precise than alternative methods such as differential approximants, but it can usefully supplement them to ascertain that the singularities identified are not artefacts.

In fluid dynamics one outstanding problem is the issue of finite-time blow-up for the three-dimensional incompressible Euler flow with smooth initial data.(10,41) For initial data having simple trigonometric polynomial form, one can determine numerically a number of coefficients of the Taylor expansion in time of the enstrophy (integral of one-half the squared vorticity). This was done for the Taylor–Green flow by Brachet *et al.*(42) (yielding 40 non-vanishing coefficients calculated with quadruple working precision) and for the Kida–Pelz flow by Pelz and Gulak(43) (yielding 16 non-vanishing coefficients having at least 40-digit precision). Because the number of coefficients is rather small, there is no consensus on what the results imply for blow-up. Such calculations have a complexity $O(N^5)$ which can however be reduced to $O(N^4)$ (up to logs) using the method of relaxed multiplication.(39) It is likely that a state-of-the-art calculation for flows with simple trigonometric polynomial initial conditions can give up to several hundred non-vanishing Taylor coefficients of the enstrophy, with a working precision of several hundred digits. Another problem which can be tackled by series analysis is the analytic structure of the two-dimensional incompressible vortex sheet (Kelvin–Helmholtz instability). It is known that an initially analytic interface will develop a singularity in its shape after a finite time. Moore has made a prediction regarding this singularity(44) which has been studied by various numerical techniques.(16,45) Again there is no consensus on the type of this singularity.

It is of course of interest to extend to several dimensions the BPH method, here presented only in the one-dimensional case. We observe that there exist multi-dimensional generalizations of the Borel transform(46–48) and that the the asymptotic interpolation method can also in principle be extended to several dimensions.(11) In several dimensions, singularities are not point-like; they reside on extended objects such as analytic manifolds and can have a much more involved structure than in one dimension. It is possible to partially reconstruct such objects using a variant of BPH. Furthermore, we note that Pólya's theorem has been extended to several complex dimensions(48) (it is then referred to as the Ivanov–Stavskiĭ theorem). Information on singularities can then in principle be obtained numerically in a way analogous to what has been done in Sec. 5. Such issues will be discussed elsewhere.

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